Vacuum energy density in arbitrary background fields

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Abstract. The vacuum expectation value of the energy-momentum tensor of a real scalar field in the presence of an arbitrary scalar background field is considered. The problem of renormalization is treated in detail. In the special case of a background field depending on one coordinate only, we give an explicit integral representation for the renormalized vacuum energy. Three explicit examples illustrate the use of this representation as well as some properties of the vacuum energy density. We find that a twice continuously differentiable background potential leads to a continuous energy density.

1. Introduction

The analysis of quantum vacuum effects can be divided into local and global investigations. The divergences play a crucial role in this analysis. They are all local. The traditional subject for investigations are manifolds with sharp boundaries (i.e. conductor boundary conditions on the surface of a metallic body). In the past few years there has been an increasing interest in more general or, in the context of applications, more realistic boundaries: penetrable walls [1, 2], soft and semihard boundaries [3]. The next step in this direction is the investigation of a general background field which can be deformed in order to model boundaries of various weakness. This is of particular interest for the study of the backreaction problem. The boundaries or background fields are classical objects whose dynamics will be influenced by the vacuum fluctuations of all the quantum fields which may be present in that background. With respect to this there is an essential difference between smooth background fields and boundary conditions. While smooth fields do have their own dynamics (i.e. a finite classical action) boundary conditions do not. The reason is simply that a deformation of a smooth background field into a sharp boundary (e.g. the deformation of a smeared out potential wall into a rectangular wall and, further, into a delta function which can be represented by specific boundary conditions) requires an infinite amount of energy. Nevertheless, sharp boundaries are capable of describing reasonable physics (e.g. the Casimir effect [4]). In view of this we consider the study of this transition as an interesting problem and emphasize the role of a local consideration. There are at least two further reasons to do so. The first reason is that some local divergences arising in such a transition disappear in a calculation which is global from the very beginning. The second reason is the necessity to deal with local quantities in the semiclassical theory of gravity when inserting vacuum polarization contribution into the right-hand side of the Einstein equations [5].

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In the present paper we consider a scalar field φ in flat $(3 + 1)$-dimensional spacetime interacting with a classical background field $\Phi(x)$. The renormalization which is known in general in terms of the corresponding heat-kernel coefficients [6] is analysed in detail, especially the role of the renormalization of a contribution to the classical action which constitutes a complete derivative is emphasized. In order to discuss the above-mentioned transition from a smooth background field to sharper boundaries we consider the special case of a background field $\Phi(x_3)$ depending on one coordinate x_3 only and perform explicit calculations for three explicit examples:

(i) Φ^2 is a piecewise quadratic function

(ii) Φ^2 is a piecewise linear function

(iii) Φ^2 is a piecewise constant function ('square wall').

Whereas the background field in the first case has a finite classical energy, in the second and third cases it does not (the kinetic energy is infinite). It is interesting to observe that in all three cases the expectation value of the energy-momentum tensor shows a divergent behaviour as a function of x_3, whereas the global energy in these cases does not contain divergences and behaves very similarly in all these cases.

For our investigations we choose the simplest model which contains all terms necessary for the renormalization. It is described by the Lagrange density

$$L = \tfrac{1}{2}\Phi(-\partial_\mu\partial^\mu - M^2 - \lambda\Phi^2)\Phi + \tfrac{1}{2}\varphi(-\partial_\mu\partial^\mu - m^2 - \lambda'\Phi^2)\varphi \qquad (1)$$

where φ denotes the quantized scalar field. Φ is a classical Φ^4-self-interacting field modelling the background interaction via the identification $V(x) = \lambda'\Phi^2$. Thus, the interaction with the external system leads to the modified equation of motion

$$(\partial_\mu\partial^\mu + m^2 + V(x))\varphi = 0 \qquad (2)$$

rather than to boundary conditions. The global vacuum energy for this model has been calculated in [7]. Here we are interested in the vacuum expectation value of the EMT of the whole system which can conveniently be represented by

$$\langle 0|T_{\mu\nu}|0\rangle = \partial_\mu\Phi\partial_\nu\Phi - \tfrac{1}{2}g_{\mu\nu}(\partial_\rho\Phi\partial^\rho\Phi - M^2\Phi^2 - \lambda\Phi^4)$$

$$+ \left[\left(\partial_{y_\mu}\partial_{x_\nu} - \frac{1}{2}g_{\mu\nu}\partial_{y_\rho}\partial_x^\rho + g_{\mu\nu}\frac{m^2}{2} + g_{\mu\nu}\frac{\lambda'}{2}\right)\frac{1}{i}G(x, y)\right]_{x=y} \qquad (3)$$

where $G(x, y)$ denotes the causal propagator $G(x, y) = i\langle 0|T\varphi(x)\varphi(y)|0\rangle$. Covariant point splitting serves as a temporary regularization.

The paper is organized as follows. In section 2 we solve the problem of renormalization by utilizing the heat-kernel-expansion of the propagator. The needed coefficients and their derivatives are calculated. In section 3 we restrict ourselves to background fields which depend on one coordinate only. We construct an integral representation of the renormalized vacuum expectation value of the EMT in terms of the corresponding one-dimensional scattering basis. The counter-terms of the preceding section are recovered. Section 4 contains three examples which allow for an explicit representation of the scattering basis in terms of special functions. We calculate numerically the energy density as well as the pressure. The behaviour of the energy density in the vicinity of points of discontinuity of the potential or its derivatives is discussed in detail. We conclude with a brief summary of the results.

2. Renormalization

The term in the square brackets in equation (3) represents the contribution of the quantum field φ to the vacuum EMT of the whole system. It diverges for coinciding arguments and

needs to be renormalized. It is expressed entirely in terms of the causal propagator. The heat-kernel expansion [5, 6, 10] provides us with the appropriate technique to isolate the counter-terms.

The heat kernel of a self-adjoint operator $\hat{D}$ satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}K(x, y|t) = \hat{D}K(x, y|t) \tag{4}$$

along with the initial condition

$$K(x, y|0) = \delta(x - y). \tag{5}$$

The corresponding Green's function is related to the heat-kernel through the equality

$$G(x, y) = i\int_0^\infty K(x, y|t)\,dt. \tag{6}$$

In the present case the operator $\hat{D}$ is given by $\hat{D} = \partial_\mu \partial^\mu + m^2 + V(x)$ and equation (4) specializes to

$$i\frac{\partial}{\partial t}K(x, y|t) = (\partial_\mu \partial^\mu + m^2 + V(x))K(x, y|t). \tag{7}$$

For $V \equiv 0$ the solution obeying initial condition (5) is explicitly known

$$K_0(x, y|t) = \frac{-i}{(4\pi t)^2}\exp\left[-i\left(m^2 t + \frac{(x - y)^2}{4t}\right)\right]. \tag{8}$$

We now represent $K(x, y|t)$ by

$$K(x, y|t) = H(x, y|t)K_0(x, y|t). \tag{9}$$

The initial condition (5) implies $H(x, y|0) = 1$. $H(x, y|t)$ is known to have the following asymptotic expansion for $t \to 0$:

$$H(x, y|t) \approx \sum_{n=0}^\infty a_n(x, y)(it)^n \qquad a_0 = 1. \tag{10}$$

Substituting the ansatz (9) into the equation of motion (7), one obtains a recurrence relation for the coefficients $a_n(x, y)$:

$$\left(\partial_\mu \frac{(x - y)^2}{2}\partial^\mu + n + 1\right)a_{n+1}(x, y) + (\partial_\mu \partial^\mu + V(x))a_n(x, y) = 0. \tag{11}$$

Taking into account the initial condition $a_0 = 1$, this relation allows for the computation of the coefficients $a_n(x, y)$ and their derivatives with respect to x and y for coinciding arguments $x = y$ to any desired order n. Subsequent differentiation of equation (11) is necessary during this process. For example, the coefficient $a_1(x, x)$ is obtained by taking the limit $y \to x$ in equation (11)

$$(n + 1)a_{n+1}(x) = -(\partial_\mu \partial^\mu a_n)(x) - V(x)a_n(x). \tag{12}$$

Then, putting $n = 0$ results in $a_1(x, x) = -V(x)$. We just display the coefficients and derivatives which contribute to the counter-terms when renormalizing $\langle 0|T_{\mu\nu}|0\rangle$

$$\begin{aligned}
a_0(x) &= 1 \\
a_1(x) &= -V(x) \\
a_2(x) &= \tfrac{1}{2}\left(\tfrac{1}{3}\partial_\mu \partial^\mu V(x) + V^2(x)\right) \\
(\partial_y^\kappa \partial_x^\nu a_1)(x) &= -\tfrac{1}{6}\partial_y^\kappa \partial_x^\nu V(x) \\
(\partial_y^\kappa \partial_x^\nu a_2)(x) &= \tfrac{1}{3}\left[\tfrac{1}{10}\partial^\kappa \partial^\nu \partial_\mu \partial^\mu V + \tfrac{1}{2}V\partial^\kappa \partial^\nu V + \tfrac{3}{4}(\partial^\kappa V)(\partial^\nu V)\right](x).
\end{aligned} \tag{13}$$

We now turn to the calculation of the Green's function. Inserting (8) and (9) into equation (6) we obtain

$$G(x, y)^s = \frac{\Gamma(1+s)\mu^{2s}}{(4\pi)^2} \sum_{n=0}^{\infty} a_n(x, y) i^{n+s} \int_0^{\infty} dt\, t^{n+s-2} e^{-i(m^2 t + (x-y)^2/4t)} \quad (14)$$

where the parameter s has been introduced as a regularization. The factor μ^{2s} ensures correct dimensions. The limit $s \to 0$ removes the regularization. Utilizing the integral representation of the second class Hankel function ($\mathrm{Im}(z) < 0$)

$$H_p^{(2)}(z) = \frac{i^{p+1}}{\pi} \int_0^{\infty} du\, u^{-(p+1)} \exp\left[-\frac{i}{2} z\left(u + \frac{1}{u}\right)\right] \quad (15)$$

the expression (14) can be cast into the compact form

$$G(x, y)^s = \frac{\Gamma(1+s)\mu^{2s}}{16\pi} \sum_{n=0}^{\infty} a_n(x, y) \left(\frac{z}{2m^2}\right)^{n+s-1} H_{n+s-1}^{(2)}(z) \quad (16)$$

with the abbreviation $z = \sqrt{m^2(x-y)^2}$. See e.g. [5, 8]. We are now in a position to calculate the contribution of the quantum field φ to the vacuum expectation value of the EMT according to equation (3). The result is ($s > 2$)

$$\langle T^{\mu\nu}(x)\rangle_0^s = \frac{\Gamma(1+s)\mu^{2s}}{16\pi^2} \sum_{n=0}^{\infty} (2m^2)^{1-n-s} \left[2^{n+s-2}\Gamma(n+s-2)a_n(x, x)m^2 g^{\mu\nu} \right.$$

$$+ 2^{n+s-1}\Gamma(n+s-1)\left[\frac{g^{\mu\nu}}{2}(m^2 + V(x))a_n(x, x) + (\partial_y^\mu \partial_x^\nu a_n)(x)\right.$$

$$\left.\left. - \frac{g^{\mu\nu}}{2} g_{\lambda\rho}(\partial_y^\lambda \partial_x^\rho a_n)(x)\right]\right]. \quad (17)$$

Here we have used the property

$$\lim_{z\to 0}(z^p H_p^{(2)}(z)) = \frac{i}{\pi} 2^p \Gamma(p) \ \forall p \qquad \mathrm{Re}(p) > 0 \quad (18)$$

when performing the limit $y \to x$. The right-hand side of equation (17) is an analytic function of s in the half plane $\mathrm{Re}(s) > 2$. It can be continued analytically to the rest of the complex plane with simple poles at $s = 0, 1, 2$ and the negative integers. When removing the regularization $s \to 0$, pole terms occur in the first three summands of expression (17) only. We denote them by $\langle T^{\mu\nu}\rangle_0^{div}$ and expand them around the point $s = 0$. Inserting the heat-kernel coefficients calculated above yields

$$\langle T^{\mu\nu}\rangle_0^{div} = \frac{1}{16\pi^2}\left[-\frac{m^4}{4} g^{\mu\nu}\left(\frac{1}{s} - 2C + \frac{1}{2} + \ln\frac{\mu^2}{m^2}\right) - \frac{m^2}{2} V(x) g^{\mu\nu}\left(\frac{1}{s} - 2C + \ln\frac{\mu^2}{m^2}\right)\right.$$

$$- \frac{1}{4} V^2(x) g^{\mu\nu}\left(\frac{1}{s} - 2C - 1 + \ln\frac{\mu^2}{m^2}\right)$$

$$+ \frac{1}{6}(-\partial^\mu \partial^\nu V + g^{\mu\nu}\partial_\rho \partial^\rho V)\left(\frac{1}{s} - 2C + \frac{1}{2} + \ln\frac{\mu^2}{m^2}\right)$$

$$+ \frac{V}{4m^2} g^{\mu\nu}\left(\frac{1}{3}\partial_\rho \partial^\rho V + V^2\right) + \frac{g^{\mu\nu}}{12}\partial_\rho \partial^\rho V$$

$$\left. + \frac{1}{m^2}\left(\partial_y^\mu \partial_x^\nu a_2 - \frac{g^{\mu\nu}}{2} g_{\lambda\rho}\partial_y^\rho \partial_x^\lambda a_2\right) + o(s)\right]. \quad (19)$$

C denotes Euler's constant $C = 0, 577 \ldots$. The structure of the pole terms in this expression is to be investigated next. The first term which is independent of the potential $V(x)$ is identified with the usual free space contribution. According to the common procedure, its removal can be interpreted as a renormalization of the cosmological constant. Remembering the definition $V(x) = \lambda' \Phi^2$, we find that the terms containing V and V^2, respectively, are proportional to the terms $-g^{\mu\nu} \frac{M^2}{2} \Phi^2$ and $-g^{\mu\nu} \frac{\lambda}{2} \Phi^4$ of the classical part of the EMT. Hence, they can be absorbed in a renormalization of the mass M and the coupling constant λ of the classical background field Φ. The term containing derivatives forms a total divergence

$$-\partial^\mu \partial^\nu V + g^{\mu\nu} \partial_\rho \partial^\rho V = \partial_\rho (-g^{\mu\rho} \partial^\nu V + g^{\mu\nu} \partial^\rho V). \tag{20}$$

The expression in the brackets on the right-hand side is antisymmetric in the indices ν and ρ. We are free to add those terms to the canonical EMT. If we choose, for instance, the expression

$$\frac{\kappa}{2} (-\partial^\mu \partial^\nu \Phi^2 + g^{\mu\nu} \partial_\rho \partial^\rho \Phi^2) \tag{21}$$

we can get rid of the remaining pole term via a renormalization of the constant κ. It should be mentioned that the total divergence term does not consequently occur in the calculation of the global vacuum energy [7].

As for the normalization conditions we demand that the contribution of the quantum field to the EMT vanishes for zero potential $V \equiv 0$. This simply refers to the removal of the constant term. The normalization condition for the remaining terms is chosen so that the EMT of the quantum system does not contain contributions proportional to the classical terms. With these conditions in hand we arrive at the following renormalization of the constants of the classical system Φ

$$M^2 \longrightarrow M^2 - \frac{m^2 \lambda'}{16\pi^2} \left(\frac{1}{s} - 2C + \ln \frac{\mu^2}{m^2} \right)$$

$$\lambda \longrightarrow \lambda - \frac{\lambda'^2}{32\pi^2} \left(\frac{1}{s} - 2C - 1 + \ln \frac{\mu^2}{m^2} \right) \tag{22}$$

$$\kappa \longrightarrow \kappa + \frac{\lambda'}{48\pi^2} \left(\frac{1}{s} - 2C + \ln \frac{\mu^2}{m^2} \right).$$

The vacuum expectation value of the EMT of the whole system (3) has been renormalized in terms of the constants of the classical system Φ. The necessity of the embedding of the quantum field φ in the classical background Φ has become evident.

3. The special case of a one-dimensional background field

In this section we restrict ourselves to the investigation of background potentials depending on one coordinate only, for instance $V(x) = V(x_3)$. We construct an integral representation for the ground state energy density as well as for the vacuum pressure. We assume that the potential decreases sufficiently fast at infinity in order for scattering theory to be applicable.

Separation of variables in the equation of motion (2) becomes possible if the background potential is one-dimensional. The solutions corresponding to the directions of unbroken translational invariance are free plane waves. It remains to consider the one-dimensional Schrödinger equation

$$[\partial_{x_3}^2 + k^2 - V(x_3)] \varphi(x_3) = 0 \tag{23}$$

with the abbreviation $k^2 = k_0^2 - k_1^2 - k_2^2 - m^2$. The k_α, $\alpha = 0, 1, 2$ are the momenta of the free directions. The full propagator $G_3(x_3, y_3)$ can be expressed through the Green's function of the one-dimensional problem (23)

$$G(x, y) = \int \frac{d^3 k_\alpha}{(2\pi)^3} \exp[ik_\alpha(x^\alpha - y^\alpha)]G_3(x_3, y_3) \qquad \alpha = 0, 1, 2 \qquad (24)$$

where $G_3(x_3, y_3)$ satisfies

$$[\partial_{x_3}^2 + k^2 - V(x_3)]G_3(x_3, y_3) = -\delta(x_3 - y_3). \qquad (25)$$

According to the theory of Sturm–Liouville, the Green's function $G_3(x_3, y_3)$ can be represented in terms of an integral basis of equation (23) which is subject to appropriate boundary conditions

$$G_3(x_3, y_3) = \frac{1}{W(\varphi_1, \varphi_2)}[\theta(y_3 - x_3)\varphi_1(y_3)\varphi_2(x_3) + \theta(x_3 - y_3)\varphi_1(x_3)\varphi_2(y_3)]. \qquad (26)$$

$W(\varphi_1, \varphi_2)$ denotes the Wronski determinant. The boundary conditions have to be chosen in such a way that equation (24) yields the causal propagator. It turns out that the correct choice is the so-called scattering basis $\varphi_1(x_3)$, $\varphi_2(x_3)$ satisfying the asymptotic conditions

$$\varphi_1(x_3) \underset{x_3 \to -\infty}{\sim} e^{ikx_3} + s_{12}e^{-ikx_3} \qquad \varphi_1(x_3) \underset{x_3 \to \infty}{\sim} s_{11}e^{ikx_3}$$

$$\varphi_2(x_3) \underset{x_3 \to -\infty}{\sim} s_{22}e^{-ikx_3} \qquad \varphi_2(x_3) \underset{x_3 \to \infty}{\sim} e^{-ikx_3} + s_{21}e^{ikx_3}. \qquad (27)$$

The vacuum expectation value of the EMT can now be expressed in terms of the scattering basis. We just have to substitute the propagator (24) together with expression (26) into the defining equation (3). Due to the symmetry of the problem, $\langle 0|T_{\mu\nu}|0\rangle$ appears to have only two independent components. In particular we find

$$\langle T^{00}(x_3)\rangle_0 = \frac{1}{2i} \int \frac{d^3 k_\alpha}{(2\pi)^3} \frac{(2k_0^2 - k^2 + V)\varphi_1\varphi_2 + \varphi_1'\varphi_2'}{W(\varphi_1, \varphi_2)} \qquad (28)$$

and

$$\langle T^{33}(x_3)\rangle_0 = \frac{1}{2i} \int \frac{d^3 k_\alpha}{(2\pi)^3} \frac{(k^2 - V)\varphi_1\varphi_2 + \varphi_1'\varphi_2'}{W(\varphi_1, \varphi_2)}. \qquad (29)$$

Of course, these quantities have not yet been renormalized. The integrals (28) and (29) are ultraviolet divergent. The counter-terms need to be isolated. This can be achieved by asymptotically expanding the integrand. However, knowledge of the asymptotic behaviour of the scattering basis which obeys the asymptotic conditions (27) becomes necessary. The problem can be resolved if the fact is taken into account that the counter-terms are of purely local nature. It is therefore sufficient to consider background potentials with compact support, i.e.

$$V(x_3) = 0 \qquad \text{if } x_3 \notin [-d_1, d_2]. \qquad (30)$$

In this case, it is possible to incorporate the conditions (27) explicitly according to

$$\varphi_1(x) = \begin{cases} e^{ikx} + s_{12}e^{-ikx} & x \leqslant -d_1 \\ \alpha_1 u(x) + \beta_1 v(x) & -d_1 \leqslant x \leqslant d_2 \\ s_{11}e^{ikx} & d_2 \leqslant x \end{cases} \qquad (31)$$

and

$$\varphi_2(x) = \begin{cases} s_{22}e^{-ikx} & x \leqslant -d_1 \\ \alpha_2 u(x) + \beta_2 v(x) & -d_1 \leqslant x \leqslant d_2 \\ e^{-ikx} + s_{21}e^{ikx} & d_2 \leqslant x \end{cases} \qquad (32)$$

where $u(x)$, $v(x)$ denotes an arbitrary basis of equation (23). We just write x instead of x_3 in what follows. We have to demand that the scattering waves φ_i and their first derivatives be continuous at the points $x = -d_1, d_2$. Thus we get a system of linear equations for the coefficients s_{ij}, α_i and β_i. This system is solved by standard methods. As a result, everything is expressed entirely in terms of the arbitrary basis $u(x)$, $v(x)$. For example, the energy density for $-d_1 < x < d_2$ is given by

$$
\langle T^{00}(x)\rangle_0^s = \frac{\mu^{2s}}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\gamma^{1+2s}} \left[\frac{\alpha_1\alpha_2}{s_{11}} ([\gamma^2 - 2k_4^2 + V]u^2(x) + u'^2(x)) \right.
$$
$$
+ \frac{\beta_1\beta_2}{s_{11}} ([\gamma^2 - 2k_4^2 + V]v^2(x) + v'^2(x))
$$
$$
\left. + \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{s_{11}} ([\gamma^2 - 2k_4^2 + V]u(x)v(x) + u'(x)v'(x)) \right] \tag{33}
$$

where a Wick rotation $k_0 \to ik_4$, $k \to i\gamma = i\sqrt{k_4^2 + k_1^2 + k_2^2 + m^2}$ has been performed. The factor γ^{-2s} serves as a regularization and is removed in the limit $s \to 0$. The basis $u(x)$, $v(x)$ can be chosen to have an appropriate asymptotic behaviour. After the Wick rotation it satisfies the equation

$$
[\partial_{x_3}^2 - \gamma^2 - V(x_3)]\varphi(x_3) = 0. \tag{34}
$$

We make an ansatz for a solution exhibiting exponential behaviour for large values of γ

$$
u(x) = e^{\gamma x} \sum_{n=0}^{\infty} u_n(x)\gamma^{-n}. \tag{35}
$$

The equation of motion yields the recurrence differential equation

$$
0 = 2\partial u_{n+1} + \partial^2 u_n - V u_n \tag{36}
$$

for the coefficients $u_n(x)$. With the initial condition $u_0 = 1$, they can be computed to any order. For instance, we find

$$
u_1(x) = \tfrac{1}{2} \int^x \mathrm{d}t\, V(t). \tag{37}
$$

A linear independent solution is obtained from $u(x)$ by substituting $\gamma \to -\gamma$. The asymptotic expansion of the energy density yields the following terms contributing to the counter-terms $(\mathrm{Re}(s) > 2)$

$$
\langle T^{00}\rangle_s^{div} = \frac{\mu^{2s}}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\gamma^{1+2s}}
$$
$$
\times \left[-2k_4^2 + \frac{k_4^2}{\gamma^2} V(x) - \frac{1}{\gamma^2}\frac{V''(x)}{4} + \frac{k_4^2}{\gamma^4}\left(\frac{1}{4}V''(x) - \frac{3}{4}V^2(x)\right) \right]. \tag{38}
$$

Analytical continuation and expansion around $s = 0$ leads to the following pole terms

$$
\langle T^{00}\rangle_s^{div} = \frac{1}{16\pi^2}\left[-\frac{1}{4}m^4\left(\frac{1}{s} + \frac{3}{2} + \ln 4\frac{\mu^2}{m^2}\right) - \frac{1}{2}m^2 V(x)\left(\frac{1}{s} - 1 + \ln 4\frac{\mu^2}{m^2}\right) \right.
$$
$$
\left. - \frac{1}{6}V''(x)\left(\frac{1}{s} - \frac{5}{3} + \ln 4\frac{\mu^2}{m^2}\right) - \frac{1}{4}V^2(x)\left(\frac{1}{s} - \frac{8}{3} + \ln 4\frac{\mu^2}{m^2}\right) + o(s) \right]. \tag{39}
$$

Thus we have recovered the counter-terms found in the preceding section. The normalization conditions explained there apply here as well. Then, the renormalization reads

$$M^2 \longrightarrow M^2 - \frac{m^2 \lambda'}{16\pi^2} \left(\frac{1}{s} - 1 + \ln 4 \frac{\mu^2}{m^2} \right)$$

$$\lambda \longrightarrow \lambda - \frac{\lambda'^2}{32\pi^2} \left(\frac{1}{s} - \frac{8}{3} + \ln 4 \frac{\mu^2}{m^2} \right) \tag{40}$$

$$\kappa \longrightarrow \kappa + \frac{\lambda'}{48\pi^2} \left(\frac{1}{s} - \frac{5}{3} + \ln 4 \frac{\mu^2}{m^2} \right).$$

The purely local dependence of the counter-terms on the potential permits us to re-express the renormalized vacuum expectation value of the EMT in terms of the scattering basis. Consequently, the validity of the following expression for the renormalized energy density can be extended to potentials with unbounded support.

$$\langle T^{00}(x) \rangle_0^{ren} = \int_0^\infty \frac{dr\, r^2}{(2\pi)^2} \left[\frac{(\gamma^2 - \frac{2}{3}r^2 + V)\varphi_1 \varphi_2 + \varphi_1' \varphi_2'}{W(\varphi_1, \varphi_2)} \right.$$

$$\left. + \frac{1}{2\gamma} \left\{ \frac{2}{3}r^2 - \frac{1}{3}\frac{r^2}{\gamma^2}V + \frac{1}{\gamma^2}\frac{1}{4}V'' - \frac{1}{3}\frac{r^2}{\gamma^4}\left(\frac{1}{4}V'' - \frac{3}{4}V^2 \right) \right\} \right]. \tag{41}$$

The vacuum pressure $\langle T^{33}(x_3) \rangle_0$ is treated in exactly the same fashion. We just list the result

$$\langle T^{33}(x) \rangle_0^{ren} = \int_0^\infty \frac{dr\, r^2}{(2\pi)^2} \left[\frac{-(\gamma^2 + V)\varphi_1 \varphi_2 + \varphi_1' \varphi_2'}{W(\varphi_1, \varphi_2)} + \frac{1}{2\gamma} \left\{ 2\gamma^2 + V(x) - \frac{1}{\gamma^2}\frac{1}{4}V^2(x) \right\} \right]. \tag{42}$$

We have found an integral representation in terms of the scattering basis for the renormalized vacuum expectation value of the EMT in the case of a background field depending on one coordinate only.

4. Examples

In this section we calculate the energy density and the vacuum pressure for three simple symmetric background potentials. These examples illustrate the typical behaviour of the energy density at points of discontinuity of the potential. In all three cases the integral basis u, v can be expressed in terms of special functions.

The first example is the square well potential given by

$$V(x) = \begin{cases} V_0 & |x| \leqslant d \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

It is discontinuous at the points $x = \pm d$. The basis u, v is conviently chosen as

$$u(x) = \exp\left(\sqrt{\gamma^2 + V_0}\, x\right) \qquad v(x) = \exp\left(-\sqrt{\gamma^2 + V_0}\, x\right). \tag{44}$$

The corresponding ground state energy density $\langle T^{00} \rangle_0$ and pressure $\langle T^{33} \rangle_0$ are shown in figure 1.

In the case of the piecewise linear potential

$$V(x) = \begin{cases} V_0 \left(1 - \frac{|x|}{d} \right) & |x| \leqslant d \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

$< T^{00} >_0 \cdot (2\pi)^2, \quad < T^{33} >_0 \cdot 10^6 \cdot (2\pi)^2$
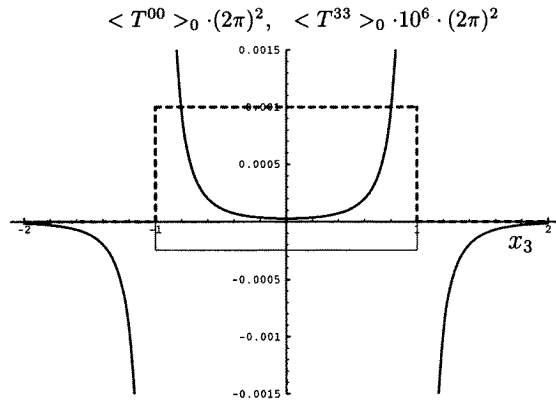

Figure 1. $\langle T^{00}\rangle_0$ (thick full curve) and $\langle T^{33}\rangle_0$ (thin full curve) for the square well potential (broken line) with $m = d = 1$ and $V_0 = 10^{-3}$.
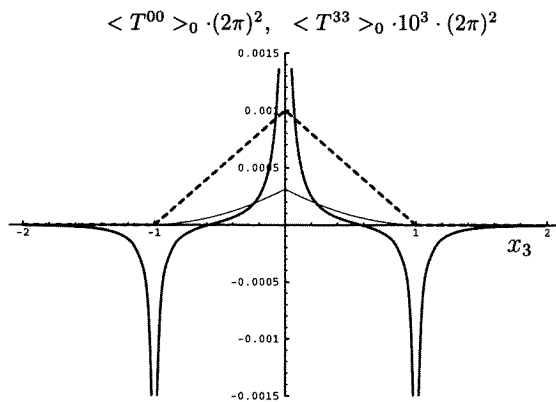
$< T^{00} >_0 \cdot (2\pi)^2, \quad < T^{33} >_0 \cdot 10^3 \cdot (2\pi)^2$

Figure 2. $\langle T^{00}\rangle_0$ (thick full curve) and $\langle T^{33}\rangle_0$ (thin full curve) for the piecewise linear potential (broken line) with $m = d = 1$ and $V_0 = 10^{-3}$

the basis u, v can be expressed in terms of Airy functions [11]

$$u(x) = Ai\left(\frac{\gamma^2}{\eta^2} \pm \eta(d - |x|)\right) \qquad v(x) = Bi\left(\frac{\gamma^2}{\eta^2} \pm \eta(d - |x|)\right) \qquad (46)$$

with the abbreviation $\eta = (\frac{|V_0|}{d})^{1/3}$. The first derivative is discontinuous at the points $x = \pm d$. The groundstate energy density and the pressure are displayed in figure 2.

The third example is the piecewise oscillatory potential

$$V(x) = \begin{cases} V_0 \left(1 - \frac{|x|}{d}\right)^2 & |x| \leqslant d \\ 0 & \text{otherwise} \end{cases} \qquad (47)$$

with its second derivative being discontinuous at $x = \pm d$. Here one finds for u, v

$$u(x) = U\left(\frac{\gamma^2}{2\eta^2}, \sqrt{2}\eta(d - |x|)\right) \qquad v(x) = U\left(\frac{\gamma^2}{2\eta^2}, -\sqrt{2}\eta(d - |x|)\right) \qquad (48)$$

where $U(k, x)$ denotes a standard solution of Kummer's equation [11]. Here, the constant η takes the value $\eta = (|V_0|/d)^{1/4}$. Figure 3 shows the corresponding energy density and the pressure.

Some remarks are in order. As expected, the ground state energy density and the pressure vanish at infinity owing to the fact that the potential V has been assumed suitable for scattering theory to apply, i.e. V decreases sufficiently fast at infinity.

$< T^{00} >_0 \cdot (2\pi)^2, \quad < T^{33} >_0 \cdot 10^3 \cdot (2\pi)^2$
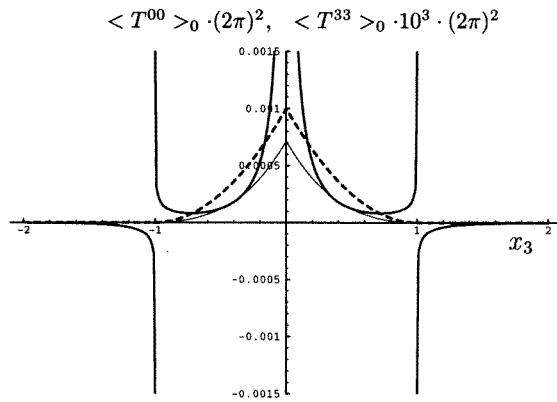

Figure 3. $\langle T^{00} \rangle_0$ (thick full curve) and $\langle T^{33} \rangle_0$ (thin full curve) for the piecewise oscillatory potential (broken line) with $m = d = 1$ and $V_0 = 10^{-3}$

The pressure $\langle T^{33}(x) \rangle_0^{ren}$ is continuous and vanishes identically outside the support of the potential V. In contrast, the energy density exhibits singularities at the points $x = 0, \pm d$ (see figures 1–3), i.e. at points of discontinuity of the potential or its derivatives. The origin of these singularities is revealed when the renormalized energy density is expanded in terms of the distance from these points. Using the asymptotic expansion (35) of the basis u, v, the expression (41) for the renormalized energy density can be split into a part converging uniformly with respect to x and a part which does not. After some involved calculations we find

$$\langle T^{00}(x) \rangle_0^{ren} = \frac{1}{8\pi^2} \left[-\frac{1}{12} \frac{e^{-2m(|x|-d)}}{(|x| - d)^2} V(d) + \frac{e^{-2m(|x|-d)}}{(|x| - d)} \left\{ \frac{1}{12} V'(d) - \frac{m}{6} V(d) \right\} \right.$$
$$\left. -\frac{2}{3} \ln(2m(|x| - d)) \left\{ \frac{1}{4} V^2(d) - \frac{1}{8} V''(d) \right\} \right] + f.T. \tag{49}$$

for $|x| > d$ and for $|x| < d$

$$\langle T^{00}(x) \rangle_0^{ren} = \frac{1}{8\pi^2} \left[\frac{1}{12} \frac{e^{-2m(d-|x|)}}{(d - |x|)^2} V(d) \right.$$
$$+ \frac{e^{-2m(d-|x|)}}{(d - |x|)} \left\{ \frac{m}{6} V(d) + \frac{1}{12} V'(d) - \frac{1}{6} V(d) \int_x^d dt\, V(t) \right\}$$
$$- \frac{2}{3} \ln(2m(d - |x|)) \left\{ \frac{1}{2} V(d) V(|x|) - \frac{1}{4} V^2(d) + \frac{1}{8} V''(d) \right.$$
$$\left. - \frac{1}{4} V'(d) \int_x^d dt\, V(t) + \frac{1}{4} V(d) \left(\int_x^d dt\, V(t) \right)^2 \right\}$$
$$\left. - \frac{e^{-2m|x|}}{|x|} \frac{1}{6} V'(0) - \frac{2}{3} \ln(2m|x|) V'(0) \frac{1}{2} \int_0^x dt\, V(t) \right] + e.T. \tag{50}$$

for $|x| < d$. With these relations in hand we are in a position to explain the behaviour of the energy density at the points $x = 0, \pm d$.

Three different terms appear which give rise to a singularity at $x = \pm d$. The term proportional to $(|x| - d)^{-2}$ is caused by the discontinuity of the potential itself. The term with $(|x| - d)^{-1}$ appears due to a jump of the first derivative, whereas the logarithmic singularity comes from a discontinuous second derivative. The examples are chosen in order to demonstrate the respective terms in leading order.

For brevity's sake, the expansions (49) and (50) have been specialized to symmetric potentials. Consequently, only an odd derivative can be discontinuous at $x = 0$, explaining

the fact that only the first derivative contributes to the singularity at $x = 0$ in equation (50). We can state that the energy density is continuous if the potential and its first and second derivative are continuous.

In [12] the behaviour of the vacuum expectation value of the EMT is studied. There, the fields are subject to sharp boundary conditions on arbitrary surfaces. It is found that the EMT diverges like ε^{-s}, $s = 1, 2, 3, 4$ on the surface, where ε measures the distance from the surface. The coefficients of each power of ε are given by certain geometric characteristics of the surface. The singularities arising in our problem may be viewed in analogy to that situation. If the potential is discontinuous, we have to demand the scattering basis φ_1, φ_2 to be continuous at those points. This is nothing else but demanding it to satisfy boundary conditions. Thus, the singularities just reflect the idealized nature of a discontinuous potential. The singularities are absent if the background potential is twice continuously differentiable.

5. Summary

The vacuum expectation value of the EMT of a real scalar field in classical background $V(x)$ has been investigated. Utilizing the heat-kernel expansion, the problem of renormalization has been solved for arbitrary scalar potentials. The necessity of the embedding in the classical system Φ has been emphasized.

In the particular case of a background potential depending on one coordinate only we were able to find an integral representation for the renormalized vacuum expectation value of the EMT in terms of the scattering basis. The ground state energy density and the vacuum pressure have been calculated explicitly for three distinct examples. The energy density has been found to diverge at points of discontinuity of the potential, its first and second derivative, respectively. A twice continuously differentiable background potential results in a continuous ground state energy density.

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